3-2 Solving Systems Algebraically

Common Core State Standards

A-REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. **Also A-REI.C.5, A-CED.A.2**

MP 1, MP 2, MP 3

Objective To solve linear systems algebraically

SOLVE IT! Getting Ready!

What whole-dollar amount of per-day sales would make it more worthwhile to work at Store B? Justify your reasoning.

HOME | JOBS | RESOURCES | HELP | LOGOUT

Search has yielded two jobs:

Store A ☐Check to apply
$35 per day plus 10% commission on all sales

Store B ☐Check to apply
$10 per day plus 18% commission on all sales

MATHEMATICAL PRACTICES

When you try to solve a system of equations by graphing, the coordinates of the point of intersection may not be obvious.

Lesson Vocabulary
- equivalent systems

Essential Understanding You can solve a system of equations by writing equivalent systems until the value of one variable is clear. Then substitute to find the value(s) of the other variable(s).

You can use the substitution method to solve a system of equations when it is easy to isolate one of the variables. After isolating the variable, substitute for that variable in the other equation. Then solve for the other variable.

Problem 1 Solving by Substitution

What is the solution of the system of equations? $\begin{cases} 3x + 4y = 12 \\ 2x + y = 10 \end{cases}$

Think
Which variable should you solve for first?
In the second equation, the coefficient of y is 1. It is the easiest variable to isolate.

Step 1
Solve one equation for one of the variables.

$$2x + y = 10$$
$$y = -2x + 10$$

Step 2
Substitute the expression for y in the other equation. Solve for x.

$$3x + 4y = 12$$
$$3x + 4(-2x + 10) = 12$$
$$3x - 8x + 40 = 12$$
$$x = 5.6$$

Step 3
Substitute the value for x into one of the original equations. Solve for y.

$$2x + y = 10$$
$$2(5.6) + y = 10$$
$$11.2 + y = 10$$
$$y = -1.2$$

The solution is $(5.6, -1.2)$.

Got It? **1.** What is the solution of the system of equations? $\begin{cases} x + 3y = 5 \\ -2x - 4y = -5 \end{cases}$

Problem 2 Using Substitution to Solve a Problem

Music **A music store offers piano lessons at a discount for customers buying new pianos. The costs for lessons and a one-time fee for materials (including music books, CDs, software, etc.) are shown in the advertisement. What is the cost of each lesson and the one-time fee for materials?**

Relate 6 · cost of one lesson + one-time fee = $300

12 · cost of one lesson + one-time fee = $480

Define Let c = the cost of one lesson.

Let f = the one-time fee.

Write $\begin{cases} 6 \cdot c + f = 300 \\ 12 \cdot c + f = 480 \end{cases}$

$6c + f = 300$ — Choose one equation. Solve for f in terms of c.

$f = 300 - 6c$

$12c + (300 - 6c) = 480$ — Substitute the expression for f into the other equation, $12c + f = 480$. Solve for c.

$c = 30$

$6(30) + f = 300$ — Substitute the value of c into one of the equations. Solve for f.

$f = 120$

Think
Which equation should you use to find f?
Use the equation with numbers that are easier to work with.

Check Substitute $c = 30$ and $f = 120$ in the original equations.

$6c + f = 300$	$12c + f = 480$
$6(30) + 120 \stackrel{?}{=} 300$	$12(30) + 120 \stackrel{?}{=} 480$
$180 + 120 \stackrel{?}{=} 300$	$360 + 120 \stackrel{?}{=} 480$
$300 = 300$ ✔	$480 = 480$ ✔

The cost of each lesson is $30. The one-time fee for materials is $120.

Got It? **2.** An online music company offers 15 downloads for $19.75 and 40 downloads for $43.50. Each price includes the same one-time registration fee. What is the cost of each download and the registration fee?

You can use the Addition Property of Equality to solve a system of equations. If you add a pair of additive inverses or subtract identical terms, you can eliminate a variable.

Problem 3 Solving by Elimination

Think

How can you use the Addition Property of Equality?
Since $-4x + 3y$ is equal to 16, you can add the same value to each side of $4x + 2y = 9$.

What is the solution of the system of equations? $\begin{cases} 4x + 2y = 9 \\ -4x + 3y = 16 \end{cases}$

$$\begin{array}{r} 4x + 2y = 9 \\ \underline{-4x + 3y = 16} \\ 5y = 25 \end{array}$$

One equation has $4x$ and the other has $-4x$. Add to eliminate the variable x.

$y = 5$ Solve for y.

$4x + 2y = 9$ Choose one of the original equations.

$4x + 2(5) = 9$ Substitute for y.

$4x = -1$ Solve for x.

$x = -\frac{1}{4}$

The solution is $\left(-\frac{1}{4}, 5\right)$.

Got It? **3.** What is the solution of the system of equations? $\begin{cases} -2x + 8y = -8 \\ 5x - 8y = 20 \end{cases}$

When you multiply each side of one or both equations in a system by the same nonzero number, the new system and the original system have the same solutions. The two systems are called **equivalent systems**. You can use this method to make additive inverses.

Problem 4 Solving an Equivalent System

What is the solution of the system of equations? $\begin{matrix} ① \\ ② \end{matrix}\begin{cases} 2x + 7y = 4 \\ 3x + 5y = -5 \end{cases}$

Think

By multiplying ① by 3 and ② by -2, the x-terms become opposites, and you can eliminate them. Add ③ and ④. Solve for y.

Write

① $2x + 7y = 4$ ③ $6x + 21y = 12$

② $3x + 5y = -5$ ④ $-6x - 10y = 10$

$11y = 22$

$y = 2$

Now that you know the value of y, use either equation to find x.

① $2x + 7(2) = 4$

$2x + 14 = 4$

$2x = -10$

$x = -5$

The solution is $(-5, 2)$.

Got It? 4. a. What is the solution of this system of equations? $\begin{cases} 3x + 7y = 15 \\ 5x + 2y = -4 \end{cases}$

b. Reasoning In Problem 4, you found that $y = 2$. Substitute this value into equation ② instead of equation ①. Do you still get the same value for x? Explain why.

Solving a system algebraically does not always provide a unique solution. Sometimes you get infinitely many solutions. Sometimes you get no solutions.

Problem 5 Solving Systems Without Unique Solutions

Think

How are the two equations in this system related?
Multiplying both sides of the first equation by −1 results in the second equation.

What are the solutions of the following systems? Explain.

A $\begin{cases} -3x + y = -5 \\ 3x - y = 5 \end{cases}$

$$0 = 0$$

Elimination gives an equation that is always true. The two equations in the system represent the same line. This is a dependent system with infinitely many solutions.

B $\begin{cases} 4x - 6y = 6 \\ -4x + 6y = 10 \end{cases}$

$$0 = 16$$

Elimination gives an equation that is always false. The two equations in the system represent parallel lines. This is an inconsistent system. It has no solutions.

Got It? 5. What are the solutions of the following systems? Explain.

a. $\begin{cases} -x + y = -2 \\ 2x - 2y = 0 \end{cases}$ **b.** $\begin{cases} 4x + y = 6 \\ 12x + 3y = 18 \end{cases}$

Lesson Check

Do you know HOW?

Solve each system by substitution.

1. $\begin{cases} 3x + 5y = 13 \\ 2x + y = 4 \end{cases}$ **2.** $\begin{cases} 2x - 3y = 6 \\ x + y = -12 \end{cases}$

Solve each system by elimination.

3. $\begin{cases} 2x + 3y = 7 \\ -2x + 5y = 1 \end{cases}$ **4.** $\begin{cases} x + 2y = -1 \\ x - y = 8 \end{cases}$

5. $\begin{cases} x - y = -4 \\ 3x + 2y = 7 \end{cases}$ **6.** $\begin{cases} 3x + 4y = 10 \\ 2x + 3y = 7 \end{cases}$

Do you UNDERSTAND? MATHEMATICAL PRACTICES

7. Vocabulary Give an example of two equivalent systems.

8. Compare and Contrast Explain how the substitution method of solving a system of equations differs from the elimination method.

9. Writing A café sells a regular cup of coffee for $1 and a large cup for $1.50. Melissa and her friends buy 5 cups of coffee and spend a total of $6. Explain how to write and solve a system of equations to find the number of large cups of coffee they bought.

Practice and Problem-Solving Exercises

MATHEMATICAL PRACTICES

A Practice

Solve each system by substitution. Check your answers.

See Problem 1.

10. $\begin{cases} 4x + 2y = 7 \\ y = 5x \end{cases}$

11. $\begin{cases} 3c + 2d = 2 \\ d = 4 \end{cases}$

12. $\begin{cases} x + 12y = 68 \\ x = 8y - 12 \end{cases}$

13. $\begin{cases} 4p + 2q = 8 \\ q = 2p + 1 \end{cases}$

14. $\begin{cases} x + 3y = 7 \\ 2x - 4y = 24 \end{cases}$

15. $\begin{cases} x + 6y = 2 \\ 5x + 4y = 36 \end{cases}$

16. $\begin{cases} t = 2r + 3 \\ 5r - 4t = 6 \end{cases}$

17. $\begin{cases} y = 2x - 1 \\ 3x - y = -1 \end{cases}$

18. $\begin{cases} r + s = -12 \\ 4r - 6s = 12 \end{cases}$

19. Money A student has some \$1 bills and \$5 bills in his wallet. He has a total of 15 bills that are worth \$47. How many of each type of bill does he have?

See Problem 2.

20. A student took 60 minutes to answer a combination of 20 multiple-choice and extended-response questions. She took 2 minutes to answer each multiple-choice question and 6 minutes to answer each extended-response question.

a. Write a system of equations to model the relationship between the number of multiple choice questions m and the number of extended-response questions r.

b. How many of each type of question was on the test?

21. Transportation A youth group with 26 members is going skiing. Each of the five chaperones will drive a van or sedan. The vans can seat seven people, and the sedans can seat five people. Assuming there are no empty seats, how many of each type of vehicle could transport all 31 people to the ski area in one trip?

Solve each system by elimination.

See Problem 3.

22. $\begin{cases} x + y = 12 \\ x - y = 2 \end{cases}$

23. $\begin{cases} x + 2y = 10 \\ x + y = 6 \end{cases}$

24. $\begin{cases} 3a + 4b = 9 \\ -3a - 2b = -3 \end{cases}$

25. $\begin{cases} 4x + 2y = 4 \\ 6x + 2y = 8 \end{cases}$

26. $\begin{cases} 2w + 5y = -24 \\ 3w - 5y = 14 \end{cases}$

27. $\begin{cases} 3u + 3v = 15 \\ -2u + 3v = -5 \end{cases}$

28. $\begin{cases} 3x + 2y = 6 \\ 3x + 3 = y \end{cases}$

29. $\begin{cases} 5x - y = 4 \\ 2x - y = 1 \end{cases}$

30. $\begin{cases} 2r + s = 3 \\ 4r - s = 9 \end{cases}$

Solve each system by elimination.

See Problems 4 and 5.

31. $\begin{cases} 4x - 6y = -26 \\ -2x + 3y = 13 \end{cases}$

32. $\begin{cases} 9a - 3d = 3 \\ -3a + d = -1 \end{cases}$

33. $\begin{cases} 2a + 3b = 12 \\ 5a - b = 13 \end{cases}$

34. $\begin{cases} 2x - 3y = 6 \\ 6x - 9y = 9 \end{cases}$

35. $\begin{cases} 20x + 5y = 120 \\ 10x + 7.5y = 80 \end{cases}$

36. $\begin{cases} 6x - 2y = 11 \\ -9x + 3y = 16 \end{cases}$

37. $\begin{cases} 2x - 3y = -1 \\ 3x + 4y = 8 \end{cases}$

38. $\begin{cases} 5x - 2y = -19 \\ 2x + 3y = 0 \end{cases}$

39. $\begin{cases} r + 3s = 7 \\ 2r - s = 7 \end{cases}$

40. $\begin{cases} y = 4 - x \\ 3x + y = 6 \end{cases}$

41. $\begin{cases} 3x + 2y = 10 \\ 6x + 4y = 15 \end{cases}$

42. $\begin{cases} 3m + 4n = -13 \\ 5m + 6n = -19 \end{cases}$

B Apply

43. Think About a Plan Suppose you have a part-time job delivering packages. Your employer pays you a flat rate of \$9.50 per hour. You discover that a competitor pays employees \$2 per hour plus \$3 per delivery. How many deliveries would the competitor's employees have to make in four hours to earn the same pay you earn in a four-hour shift?

- How can you write a system of equations to model this situation?
- Which method should you use to solve the system?
- How can you interpret the solution in the context of the problem?

Solve each system.

44. $\begin{cases} 5x + y = 0 \\ 5x + 2y = 30 \end{cases}$

45. $\begin{cases} 2m = -4n - 4 \\ 3m + 5n = -3 \end{cases}$

46. $\begin{cases} 7x + 2y = -8 \\ 8y = 4x \end{cases}$

47. $\begin{cases} 2m + 4n = 10 \\ 3m + 5n = 11 \end{cases}$

48. $\begin{cases} -6 = 3x - 6y \\ 4x = 4 + 5y \end{cases}$

49. $\begin{cases} \frac{x}{3} + \frac{4y}{3} = 300 \\ 3x - 4y = 300 \end{cases}$

50. $\begin{cases} 0.02a - 1.5b = 4 \\ 0.5b - 0.02a = 1.8 \end{cases}$

51. $\begin{cases} 4y = 2x \\ 2x + y = \frac{x}{2} + 1 \end{cases}$

52. $\begin{cases} \frac{1}{2}x + \frac{2}{3}y = 1 \\ \frac{3}{4}x - \frac{1}{3}y = 2 \end{cases}$

53. Error Analysis Identify and correct the error shown in finding the solution of $\begin{cases} 3x - 4y = 14 \\ x + y = -7 \end{cases}$ using substitution.

$x + y = -7$
$y = -7 - x$

$3x - 4y = 14$
$3x - 4(-7 - x) = 14$
$3x - 28 - 4x = 14$
$-x - 28 = 14$
$x = -42$

$y = -7 - (-42)$
$y = 35$

54. Break-Even Point Jenny's Bakery sells carrot muffins at \$2 each. The electricity to run the oven is \$120 per day and the cost of making one carrot muffin is \$1.40. How many muffins need to be sold each day to break even?

55. Open-Ended Write a system of equations in which both equations must be multiplied by a number other than 1 or −1 before using elimination. Solve the system.

STEM **56. Chemistry** A scientist wants to make 6 milliliters of a 30% sulfuric acid solution. The solution is to be made from a combination of a 20% sulfuric acid solution and a 50% sulfuric acid solution. How many milliliters of each solution must be combined to make the 30% solution?

57. Writing Explain how you decide whether to use substitution or elimination to solve a system.

58. The equation $3x - 4y = 2$ and which equation below form a system with no solutions?

(A) $2y = 1.5x - 2$
(B) $2y = 1.5x - 1$
(C) $3x + 4y = 2$
(D) $4y - 3x = -2$

For each system, choose the method of solving that seems easier to use. Explain why you made each choice. Solve each system.

59. $\begin{cases} 3x - y = 5 \\ y = 4x + 2 \end{cases}$

60. $\begin{cases} 2x - 3y = 4 \\ 2x - 5y = -6 \end{cases}$

61. $\begin{cases} 6x - 3y = 3 \\ 5x - 5y = 10 \end{cases}$

Challenge

62. **Entertainment** In the final round of a singing competition, the audience voted for one of the two finalists, Luke or Sean. Luke received 25% more votes than Sean received. Altogether, the two finalists received 5175 votes. How many votes did Luke receive?

STEM 63. **Weather** The equation $F = \frac{9}{5}C + 32$ relates temperatures on the Celsius and Fahrenheit scales. Does any temperature have the same number reading on both scales? If so, what is the number?

Find the value of a that makes each system a dependent system.

64. $\begin{cases} y = 3x + a \\ 3x - y = 2 \end{cases}$

65. $\begin{cases} 3y = 2x \\ 6y - a - 4x = 0 \end{cases}$

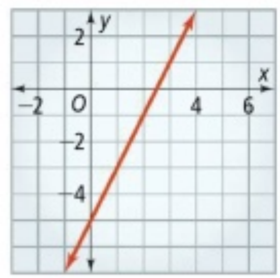
66. $\begin{cases} y = \frac{x}{2} + 4 \\ 2y - x = a \end{cases}$

Standardized Test Prep

GRIDDED RESPONSE

SAT/ACT

67. What is the slope of the line at the right?



68. What is the x-value of the solution of $\begin{cases} x + y = 7 \\ 3x - 2y = 11 \end{cases}$?

69. Solve $9(x + 7) - 6(x - 3) = 99$. What is the value of x?

70. Georgia has only dimes and quarters in her bag. She has a total of 18 coins that are worth \$3. How many more dimes than quarters does she have?

71. The graph of $g(x)$ is a horizontal translation of $f(x) = 2|x + 1| + 3$, 5 units to the right. What is the x-value of the vertex of $g(x)$?

Apply What You've Learned

MATHEMATICAL PRACTICES
MP 2

In the Apply What You've Learned in Lesson 3-1, you created a system of equations and solved the system graphically. Now, you will start to create a system of three linear equations to represent the criteria in the problem on page 133.

a. Write an equation that models the distances of all parts of the triathlon.

b. Write an equation modeling the relationship between the distances of the swim and the run of the triathlon.

c. Use substitution to write a new equation that models the distance of all parts of the triathlon, and solve it for the distance for the bicycle ride.

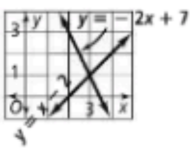
3. 2 pens; 4 pencils **4.** No; an independent system has a unique solution whereas an inconsistent system has no solution.

5. Answers may vary. Sample: $\begin{cases} y = 2x + 1 \\ y = 2x - 3 \end{cases}$

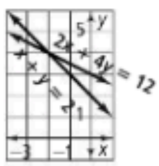
6. Independent; if the slope of one equation is the negative reciprocal of the slope of the other equation, the lines are perpendicular and intersect at a unique point.

Exercises 7–11. How solutions are determined may vary (graphing or using a table).

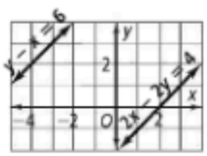
7. (3, 1)



9. (−2, 4)



11. no solution



13. 2 small; 4 large

15. Models may vary. Sample: Use 0 for 1970.

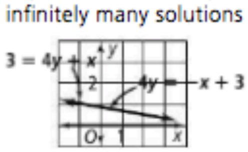
$\begin{cases} y = 0.22x + 67.5 \\ y = 0.15x + 75.507 \end{cases}$

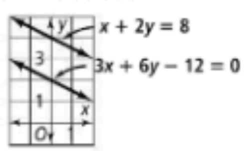
Around 2085, the quantities will be equal.

17. dependent **19.** inconsistent **21.** independent

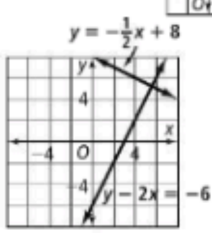
23. dependent **25.** independent **27.** dependent

29. infinitely many solutions **31.** no solution



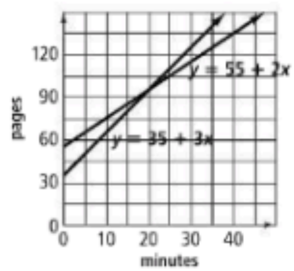


33. $\left(\frac{28}{5}, \frac{26}{5}\right)$



35. inconsistent **37.** inconsistent

39.



After 20 min you and your friend will have read the same number of pages.

41. My friend did not extend the table of values far enough. Scrolling down will show that when $x = 4$, then $Y1 = Y2 = -2$. So, the system has a solution, (4, −2).

43. An independent system has one solution. The slopes are different, but the *y*-intercepts could be the same. An inconsistent system has no solution. The slopes are the same and the *y*-intercepts are different. A dependent system has an infinite number of solutions. The slopes and *y*-intercepts are the same. **45.** sometimes **47.** never

49. Answers may vary. Sample: $5x + 2y = 5$

51. They are equivalent eqs. **53.** A **55.** C

Lesson 3-2 pp. 142–148

Got It? 1. (−2.5, 2.5) **2.** \$.95 per download; \$5.50 one-time registration fee **3.** (4, 0) **4. a.** (−2, 3) **b.** Yes; the solution (−5, 2) is a solution to both eqs. in the system, so substituting $y = 2$ into either equation will result in $x = -5$. **5. a.** no solution; The eq. is always false. **b.** infinite number of solutions; The eq. is always true.

Lesson Check 1. (1, 2) **2.** (−6, −6) **3.** (2, 1)

4. (5, −3) **5.** $\left(-\frac{1}{5}, \frac{19}{5}\right)$ **6.** (2, 1)

7. Answers may vary. Sample:

$\begin{cases} 4x - 3y = -2 \\ 3x - 2y = -1 \end{cases}$

$\begin{cases} -8x + 6y = 4 \\ 9x - 6y = -3 \end{cases}$

8. In the substitution method of solving a system of equations, you first solve one equation for one of the variables. Then substitute for this variable in the other equation and solve for the other variable. In the elimination method, you create an equivalent system of equations that contain a pair of additive inverses so that you can eliminate one variable and solve for the remaining variable.

9. Let r = number of regular cups of coffee and c = number of large cups of coffee. First, $r + c = 5$: because a total of 5 cups of coffee were purchased. Second, $r + 1.5c = 6$: because each regular cup of coffee is \$1, each large cup is \$1.50, and the total spent is \$6. Then, solve the system of equations using elimination by subtracting the first equation from the second to eliminate r and solve for c. $c = 2$; 2 large cups

Exercises **11.** $(-2, 4)$ **13.** $(0.75, 2.5)$ **15.** $(8, -1)$ **17.** $(-2, -5)$ **19.** seven \$1-bills; eight \$5-bills **21.** 3 vans and 2 sedans **23.** $(2, 4)$ **25.** $(2, -2)$ **27.** $(4, 1)$ **29.** $(1, 1)$ **31.** infinite number of solutions; $\{(x, y) | -2x + 3y = 13\}$ **33.** $(3, 2)$ **35.** $(5, 4)$

37. $\left(\frac{20}{17}, \frac{19}{17}\right)$ **39.** $(4, 1)$ **41.** no solution

43. 10 deliveries **45.** $(4, -3)$ **47.** $(-3, 4)$

49. $(300, 150)$ **51.** $(0.5, 0.25)$

53. Error in 5th line: $-4(-7 - x) = 28 + 4x$ not $-28 - 4x$; Lines 5–9 should be: $3x + 28 + 4x = 14$; $7x = -14$; $x = -2$; $y = -7 - (-2)$; $y = -5$

55. Answers may vary. Sample:

$$\begin{cases} -3x + 4y = 12 \\ 5x - 3y = 13 \end{cases}; (8, 9)$$

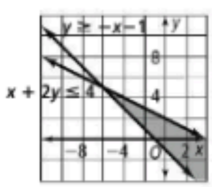
57. In determining whether to use substitution or elimination to solve an equation, look at the equations to determine if one is solved or can be easily solved for a particular variable. If that is the case, substitution can easily be used. Otherwise, elimination might be easier. **59.** Substitution; the second equation is solved for y; $(-7, -26)$ **61.** Elimination; substitution would be difficult since no coefficient is 1 in the original system. Dividing the first equation by 3 and dividing the second equation by 5 results in an equivalent system where y would be eliminated from the system if the equations were subtracted; $(-1, -3)$

63. yes; -40 degrees **65.** 0 **67.** 2 **69.** 6 **71.** 4

Lesson 3-3 pp. 149–155

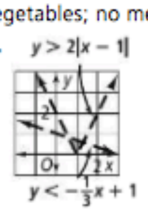
Got It? **1.** $(4, 1)$, $(5, 0)$, $(6, 0)$, $(7, 0)$

2.



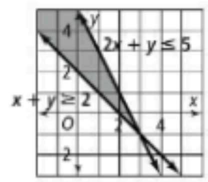
3. 5 meats and no vegetables; 4 meats and 1 or 2 vegetables; 3 meats and 2, 3, or 4 vegetables; 2 meats and 3, 4, 5, or 6 vegetables; 1 meat and 4 – 8 vegetables; no meat 5 – 10 vegetables

4.



Lesson Check

1.



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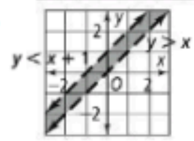
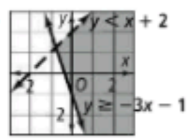


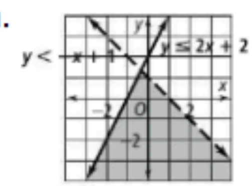
3.



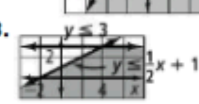
4. 0 h TV and 1, 2, or 3 h football; 1 h TV and 1 or 2 h football; or 2 h TV and 1 h football **5.** Intersection; the solution of two inequalities is the overlap or the intersection of the graphs of the individual inequalities. **6.** The graphical solution of a system of inequalities consists of the overlap or intersection of the individual half-planes and corresponding boundary lines (either dotted or solid). The graphical solution of a system of equations includes only the intersection of the lines, not the half-planes. **7.** For each inequality, the wrong half-plane has been shaded. The half-plane below $y = \frac{1}{2}x - 1$ should be shaded and the half-plane above $y = -3x + 3$ should be shaded. Also, both boundary lines should be dashed because the inequalities are $<$ and $>$.

Exercises **9.** $(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(0, 4)$, $(0, 5)$, $(0, 6)$, $(0, 7)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1,5)$, $(1, 6)$, $(2, 0)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 3)$, $(3, 4)$

11.



13.



15.